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Gauge theory, topological strings, and S-duality

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Abstract: We offer a derivation of the duality between the topological U(1) gauge theory on a Calabi-Yau 3-fold and the topological A-model on the same manifold. This duality was conjectured recently by Iqbal, Nekrasov, Okounkov, and Vafa. We deduce it from the S-duality of the IIB superstring. We also argue that the mirror version of this duality relates the topological B-model on a Calabi-Yau 3-fold and a topological sector of the Type IIA Little String Theory on the same manifold.

Keywords: D-branes, String Duality.

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1. Introduction

Recently an interesting connection has been found between Gromov-Witten invariants of a Calabi-Yau 3-fold X and a topologically twisted noncommutative U(1) gauge theory on X [1]. The connection is that the generating function of the Gromov-Witten invariants (in other words, the all-genus partition function of the A-model for X) coincides with the partition function of the topological U(1) gauge theory. According to ref. [1], the A-model string coupling λ is related to the theta-angle of the gauge theory as follows:

$$e^{-\lambda} = -e^{i\theta} \, . \tag{1.1}$$

The path integral of the A-model localizes on holomorphic maps (called holomorphic instantons) from the worldsheet to the target Calabi-Yau X. The weight of a holomorphic instanton depends on its symplectic area. This theory is "topological" in the sense that for a fixed symplectic form it does not depend on the choice of a Kähler metric, i.e. it is a symplectic invariant. On the other hand, the path-integral of the gauge theory localizes on solutions of the hermitean Yang-Mills equations (HYM) deformed by terms depending on scalars. In a U(1) gauge theory, these equations do not have interesting nonsingular solutions, but if one makes X noncommutative, nontrivial smooth solutions exist, which look like four-dimensional instantons wrapping holomorphic curves in X. This gauge theory is also "topological", in the sense that its partition function is a symplectic invariant. This can be made explicit by writing the action of the gauge theory as a sum of

$$\frac{1}{2g_s} \int \omega \wedge F \wedge F + \frac{i\theta}{6(2\pi)^3} \int F \wedge F \wedge F \, . \tag{1.2}$$

and BRST-exact terms. Thus the gauge-theory partition function depends only on $g_s^{-1}\omega$ and θ. As mentioned above, θ corresponds to the genus counting parameter of the dual topological string, while the combination $g_s^{-1}\omega$ is mapped by the duality to the symplectic form of the A-model. In ref. [1] the coincidence of the A-model partition function and the gauge-theory partition function has been verified in the limit $\omega \to \infty$ (i.e. X is replaced by flat space), and more generally for arbitrary noncompact toric Calabi-Yau manifolds. It is not clear how to extend the arguments of ref. [1] to compact Calabi-Yau manifolds. In ref. [2] the conjecture has been reformulated by replacing solutions of HYM equations on the noncommutative deformation of X with ideal sheaves on X. This reformulation is especially useful for compact X, since for such X it is not completely clear what is meant by a noncommutative deformation of the gauge theory.

Even more recently, it was noted that the duality of ref. [1] should follow from the S-duality of the Type IIB superstring [3]. In this note we develop further this idea and propose a simple physical picture explaining the coincidence of the partition functions.

2. Embedding into superstring theory

In order to embed the 6d topological gauge theory into Type IIB string theory, one can take a single Euclidean D5 brane wrapped on the Calabi-Yau manifold X. This can be thought of as an instanton in the remaining four noncompact dimensions. The theta-angle of the gauge theory on the D5-brane is identified with the RR 0-form C_0 (which is constant for the D5-brane solution). It is well-known that the low-energy action of the gauge theory on the D5 worldvolume is identical to the action of the topological gauge theory in $d = 6$ obtained by gauge-fixing the action eq. (1.2) [4, 5]. Thus the partition functions of the two theories coincide. They are both functions of θ and $g_s^{-1}\omega$. To make the gauge theory noncommutative, one can turn on a flat NS B-field on X, as usual [8, 9].

One may ask about the significance of the partition function of the D5 brane from the point of view of the effective field theory in four dimensions. Type IIB superstring compactified on X gives rise to $N = 2$ supergravity which contains $h^{2,1}(X)$ vector multiplets, $h^{1,1} + 1$ hypermultiplets, and the gravity multiplet. Out of $h^{1,1} + 1$ hypermultiplets, the first $h^{1,1}$ come from the Kähler moduli of X and their superpartners. The last one, called the universal hypermultiplet, contains the dilaton, the dual of the NS B-field (the axion), the RR 0-form C_0, and the dual of the RR 2-form C_2. These modes are constant along X, i.e. they come from $h^{0,0}(X)$.

There are two kinds of F-terms in $N = 2$ supergravity: the ones depending only on the vector multiplets and the gravity multiplet (the latter is described by a chiral Weyl superfield), and the ones depending only on the hypermultiplets. Since the dilaton sits in a hypermultiplet, the former F-term cannot receive quantum corrections and is tree-level exact. In fact it can be computed in terms of the all-order partition function of the B-model on X [6, 7], where the genus expansion of the B-model corresponds to the expansion of the F-terms in powers of the Weyl superfield. On the other hand, the hypermultiplet F-terms can receive corrections, both perturbative and nonperturbative. Nonperturbative corrections come from Euclidean D-branes or NS5-branes wrapping X. One expects that the contribution of the D5-instanton to the F-terms is proportional to the D5 partition function.

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3. The derivation of the duality

The main object of interest will be the partition function of Type IIB string theory in a background with a D5-brane instanton, in the limit when the string coupling g_s goes to zero. Closed-string degrees of freedom decouple in this limit and can be regarded as a fixed classical background. On the other hand, the effective coupling of the gauge theory on the D5-brane is ω/g_s, so we will also let ω go to zero, with the ratio ω/g_s fixed. The limit $\omega \to 0$ is the opposite of the zero-slope limit, where the 6d super-Yang-Mills action is applicable. However, we will argue below, using S-duality and supersymmetric nonrenormalization theorems, that higher-derivative terms in the D5-brane action do not contribute to the partition function, so in the limit we are considering the open-string partition function coincides with the partition function of the topological gauge theory with an action eq. (1.2).

We now propose a different way to compute the same partition function. First we perform S-duality which turns the D5-brane into an NS5-brane wrapped on the same Calabi-Yau. From the point of view of $N = 2$ supergravity, the D5-instanton is a singular field configuration involving the fields of the universal hypermultiplet. After S-duality, the situation is slightly better: the field configuration corresponding to the NS5-brane instanton is nonsingular, because of the famous "throat behavior" of the metric, but the dilaton grows without bound as one goes down the throat. Therefore we compactify one direction transverse to the NS5-brane and perform a further T-duality on it. This transformation turns the NS5-brane into a Kaluza-Klein monopole. In other words, we end up with Type IIA string theory on the direct product of X and the Taub-NUT space Y.[1] After T-duality the string coupling becomes constant (and large). It is also important to determine the mapping of the RR 0-form. It is easy to check that a constant RR 0-form is mapped by S and T-dualities to the RR 1-form C_1 with an anti-self-dual field-strength. Explicitly, it is given by

$$C_1 = aV^{-1}(dt + \omega_i dx^i) \,.$$

Here $V = 1 + 1/r$ is a harmonic function on $\mathbb{R}^3$ which appears in the Gibbons-Hawking ansatz for the metric, and ω_i satisfies curl $\vec{\omega} =$ grad V. This is a unique U(1) instanton on the Taub-NUT space. The overall factor a parametrizes the asymptotic Wilson loop of C_1 in the t-direction. S and T-dualities identify

$$a = C_0 \left(\frac{1}{g_s^2} + C_0^2 \right)^{-1} \,,$$

where C_0 is the asymptotic value of the RR 0-form, and g_s is the asymptotic string coupling in the original Type IIB background.

[1] If $C_0 \neq 0$, then this statement is precisely true only in the limit $g_s \to 0$. For general g_s the Type IIA geometry is a warped product, i.e. one has a fiber bundle with fiber X, such that the metric restricted to any fiber is conformally related to a fixed Calabi-Yau metric, and the conformal factor depends on the base coordinates. We are interested in the limit where C_0 is fixed and g_s goes to zero. In this limit there is no warping (the conformal factor becomes constant).

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Now note that C_1 in Type IIA superstring compactification is the graviphoton. Thus the composition of S and T dualities maps the D5 instanton to a smooth four-dimensional Type IIA background with an anti-self-dual (ASD) metric (the Taub-NUT metric) and an ASD graviphoton field-strength. The partition function of this background is the exponential of the quantum effective action of the $N=2$ $d=4$ supergravity.

The full supergravity action is known only to leading order in the derivative expansion. However, for anti-self-dual metric and the anti-self-dual graviphoton the exact action can be expressed in terms of the partition function of the topological A-model of the Calabi-Yau [6, 7], at least if one restricts to terms polynomial in the curvatures. Namely, for Type IIA theory, only gravitational F-terms of the form

$$\int R_- \wedge R_- (\tilde{g}_s^2 F_-)^{2g-2} d^4x \tag{3.1}$$

contribute. Here R_- is the ASD part of the Riemann tensor (in the string frame), F_- is the ASD part of $F_2 = dC_1$, and $\tilde{g}_s$ is the Type IIA string coupling. If one passes to the Einstein frame, then this terms becomes independent of $\tilde{g}_s$, and since $\tilde{g}_s$ sits in the universal hypermultiplet, which cannot appear in gravitational F-terms, we see that the coefficient of this term should be $\tilde{g}_s$-independent. This result follows from constraints of supersymmetry, and therefore is valid nonperturbatively. This is very important for us, since we are interested in the limit $g_s \to 0$, which corresponds to the Type IIA string coupling

$$\tilde{g}_s = g_s \left(\frac{1}{g_s^2} + C_0^2 \right)$$

going to infinity.

According to refs. [6, 7], the coefficient of the F-term eq. (3.1) is proportional to F_g, the genus-g partition function of the A-model on X. For our purposes, the precise coefficient will not be very important. What is important is that $F_- \sim a$, and therefore the F-term eq. (3.1) is proportional to

$$(\tilde{g}_s^2 F_-)^{2g-2} \sim (\tilde{g}_s^2 a)^{2g-2} = \left(C_0 \left(1 + g_s^2 C_0^2 \right) \right)^{2g-2} .$$

Therefore in the limit $g_s \to 0$ the expansion of the Type IIA free energy in powers of $\tilde{g}_s^2 F_-$ is the same as the expansion of the D5-brane free energy in powers of C_0.

The string partition function F_g is a function of the Kähler form $\tilde{\omega}$ of X. We can relate it to the Kähler form before S and T-dualities. Under S-duality the Calabi-Yau metric gets multiplied by

$$\sqrt{\frac{1}{g_s^2} + C_0^2} \, ,$$

while under T-duality in a transverse direction it remains unchanged. Thus the Type IIA Kähler form $\tilde{\omega}$ is related to the Kähler form ω in the original set-up (with a D5-brane) by

$$\tilde{\omega} = \frac{\omega}{g_s} \sqrt{1 + g_s^2 C_0^2} \, .$$

In the limit $g_s \to 0$, $\omega \to 0$ the Type IIA Kähler form $\tilde{\omega}$ tends to a definite limit $g_s^{-1}\omega$, which is precisely the effective coupling of the topological gauge theory.

Now we equate the C_0-dependent terms in the free energy of the original Type IIB background (with a D5-brane wrapped on X) and the free energy of the effective field theory obtained by compactifying Type IIA string on X and turning on R_- and F_-. Taking into account the identifications of the parameters of the Type IIA and Type IIB backgrounds, we come to the following conclusion. Let θ_0 be the value of the θ-angle on the D5-brane for vanishing C_0. (It is usually assumed that $\theta_0 = 0$, but we will allow for a more general possibility). Let us expand the free energy of the topological gauge theory on X in powers of $\theta - \theta_0$, where θ_0 is the value of the θ-angle for vanishing C_0:

$$\log Z_{D5} = \sum_{g=1}^{\infty} (\theta - \theta_0)^{2g-2} R_g \, .$$

The coefficients R_g are functions of $g_s^{-1}\omega$. Then for $g > 1$ we must have

$$R_g \left(\frac{\omega}{g_s} \right) = b(g) F_g(\tilde{\omega}) \, ,$$

where $\tilde{\omega} = g_s^{-1}\omega$, and b_g is some g-dependent number.

Note that we only compared C_0-dependent parts of the free energies. This is the reason we restricted the range of g to $g > 1$. Thus in essence we are comparing the contributions to the free energy which are nonperturbative from the gauge theory viewpoint.

It remains to argue that the partition function of the D5-brane can be computed using the topological gauge theory with the action eq. (1.2). First of all, we expect that quantum corrections contribute only to the BRST exact terms and can be ignored. As for stringy tree-level corrections, they are of order g_s^{-1}, but have fewer powers of ω than the first term in eq. (1.2) (the net power of ω can be negative). In the limit we are considering, such terms would blow up, and then R_g would not have a well-defined limit. On the other hand, $F_g(\tilde{\omega})$ does have a well-defined limit, which means that higher-derivative terms may contribute only to the BRST-exact terms in the gauge theory action.

One can determine the numerical coefficients b_g, as well as θ_0, by computing both R_g and F_g in some particular case and comparing them. In ref. [1] this has been done for $X = \mathbb{C}^3$, and it was found that the topological string coupling and the θ-angle are related by eq. (1.1). This means that

$$R_g = (-1)^{g-1} F_g \, , \qquad \theta_0 = \pi \, .$$

This seems to suggest that for vanishing C_0 the theta-angle of the D5-brane theory is π, rather than 0. Note that $\theta = \pi$ does not break parity-invariance of the worldvolume theory, so there is no obvious contradiction here.

4. Discussion

We showed that S-duality of Type IIB string theory implies the conjecture of ref. [1]. An interesting question is the role of noncommutativity of the D5 worldvolume. In the gauge theory computation, it serves as a regulator which gives U(1) instantons (i.e. D-strings) finite size. From the string theory viewpoint, noncommutativity comes from the

NS B-field [8, 9]. From the viewpoint of the worldvolume theory of the D-strings, it is a Fayet-Iliopoulos term which lifts the Coulomb branch on the D-string worldvolume theory. Ordinarily, this Coulomb branch describes the motion of D-strings in the directions transverse to the D5-brane. Thus turning on noncommutativity makes D-strings "stick" to the D5-brane. Upon S-duality, the NS B-field turns into a (flat) RR 2-form C_2. Thus one expects that a flat C_2 makes the F-strings "stick" to the Type IIB NS5-brane. This may seem strange, since flat RR fields usually do not have any effect on F-strings. However, this is only true for vanishing NS field H_3. In general, $C_2 \wedge H_3$ serves as a source for the RR 5-form flux $F_5 = dC_4$, which can affect F-strings. It would be interesting to understand this effect in detail. Here we only make the following simple observation: the fact that flat B-field gives D-strings a finite size can be explained after S-duality in terms of the Myers effect [10, 11]. Namely, the RR 5-form flux makes F-strings expand into D3-branes stuck to the D5 worldvolume. It appears that this effect is responsible for the "sticking" of the F-strings to the NS5 worldvolume.

Another interesting topic is the mirror version of this duality. This issue has been raised and discussed in ref. [3]. We would like to point out that the answer to this question is essentially contained in a paper by Dijkgraaf, Verlinde, and Vonk [12]. The mirror statement is that the all-order B-model partition function on X computes the partition function of the Type IIA NS5-brane wrapped on X. The topological string coupling is dual to the expectation value of the RR 3-form on X (which is proportional to the holomorphic 3-form on X). The derivation of this duality in ref. [12] is almost the same as above: one starts with a Type IIA NS5-brane on X, performs T-duality, and ends up with a IIB Kaluza-Klein monopole wrapped on X. Then one identifies the gravitational F-terms in Type IIB string evaluated on a Kaluza-Klein monopole with the free energy of the Type IIA NS5-brane. One difference with the above derivation is that one does not have to appeal to S-duality. Another difference is that we have no independent way of computing the quantum partition function of the Type IIA NS5-brane. Type IIA NS5-brane in the limit $\tilde{g}_s \to 0$ is described by Little String Theory, which loosely speaking describes self-dual strings. The results of ref. [12] show that the topological sector of the Little String Theory is equivalent to the topological string theory of type B.

Nekrasov, Ooguri, and Vafa proposed [3] that the B-model partition function of X is dual to the partition function which "counts" special lagrangian submanifolds in X. According to this conjecture, the topological string coupling is dual to the expectation value of the RR 3-form. Thus this proposal seems to be closely related to the results of ref. [12] relating Type IIA LST on X and the B-model on X. One may speculate that the Type IIA LST admits some sort of BPS membranes, which can be thought of as Euclidean D2-branes stuck to the NS5 worldvolume, and that the path-integral of the LST localizes on Euclidean membranes wrapping special lagrangian 3-cycles in X. By analogy with the case of D5-branes, one expects that such BPS membranes exist as nonsingular solutions after one regularizes the LST, by turning on a suitable flat RR form (the analogue of the B-field). Since X is simply connected, the only candidate RR form is the 3-form C_3. It would be very interesting to prove or disprove these conjectures.

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